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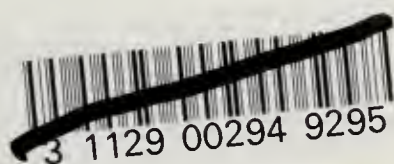
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Conditions for the Distributivity of Multiplication
with Respect to Set Addition and their Effect on
the Design of Array Multipliers

by

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1. Introduction

Multiplication is not necessarily distributive with respect to set addition. For example, using the digit sets $\{\bar{1}, 0\}$, $\{0, 1\}$, and $\{\bar{1}, 0, 1\}$, the relationship for which addition is performed first is

$$[\{\bar{1}, 0\} + \{0, 1\}] \times \{\bar{1}, 0, 1\} = \{\bar{1}, 0, 1\} \times \{\bar{1}, 0, 1\} = \{\bar{1}, 0, 1\}.$$

When multiplication is performed first, the result is

$$[\{\bar{1}, 0\} \times \{\bar{1}, 0, 1\}] + [\{0, 1\} \times \{\bar{1}, 0, 1\}] = \{\bar{1}, 0, 1\} + \{\bar{1}, 0, 1\} = \{\bar{2}, \bar{1}, 0, 1, 2\}.$$

Although the boundary between the existence of distributivity and the lack of it appears to become increasingly hard to define as the cardinality of the digit sets increases, a sufficient number of general conditions can be derived to provide guidance for the design of array multipliers.

A second set of simple examples of array multipliers indicates the nature of the problem in a practical situation. Consider first the multiplication of the 3 bit positive multiplicand $4x_2 + 2x_1 + x_0$ by the 3 bit positive multiplier $4y_2 + 2y_1 + y_0$, with the multiplier and multiplicand each ranging from 0 to +7. The product set will range, as expected, from 0 to 49, and may be formed by summing the properly weighted outputs of a 3×3 array of elementary multipliers, which are AND gates, as indicated in Figure 1.1.

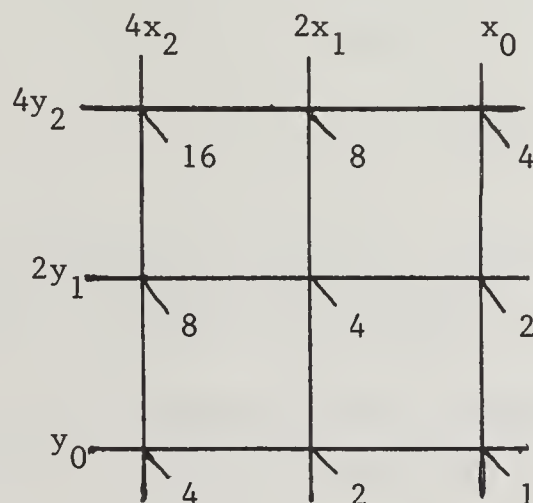


Figure 1.1 Weights for a 3×3 positive array of AND gates

It is easy to verify that the maximum sum is $16 + 2(8) + 3(4) + 2(2) + 1 = 49$ as expected.

Alternatively, consider the 3×3 array multiplier when a twos (or radix) complement representation is employed, i.e. the multiplicand is $-4x_2 + 2x_1 + x_0$ and the multiplier is $-4y_2 + 2y_1 + y_0$. Each operand ranges from -4 to $+3$, hence the product set should range from -12 to $+16$. Consider Figure 1.2, indicating the weights for the 3×3 AND gate array.

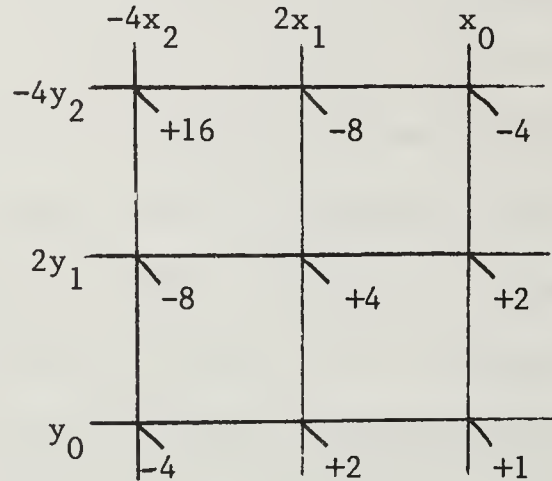


Figure 1.2 Weights for a 3×3 radix complement array of AND gates.

The summing network must be able to accomodate values ranging from the most negative value $-8 + (-8) + (-4) + (-4) = -24$ to the most positive value $16 + 4 + 2 + 2 + 1 = 25$. Obviously, distributivity does not apply.

It is the purpose of this paper to derive some general conditions for distributivity and indicate how they may be employed for the efficient design of array multipliers when negative operands are involved.

2. Notation and Terminology

A digit set, designated by $\triangle_{\delta}^{\omega}$, is characterized by two integer parameters, the diminished cardinality δ and the offset ω . The digit set is a sequence of consecutive integers, including 0, ranging in value from $-\omega$ to $+(\delta-\omega)$. A digit set is called normalized if $\omega = 0$, is called sym-

metric if $\delta = 2\omega$, (hence δ is even), and is called negative normalized if $\delta = \omega$. For example, if $\delta = 2$, the digit set $\{0,1,2\}$, with $\omega = 0$, is normalized; the digit set $\{\bar{1},0,1\}$, with $\omega = 1$, is symmetric; and the digit set $\{\bar{2},\bar{1},0\}$ with $\omega = 2$, is negative normalized.

This terminology was chosen because δ and ω are preserved under set addition. For example, if

$$\triangle_{\delta_3}^{\omega_3} \leftarrow \triangle_{\delta_1}^{\omega_1} + \triangle_{\delta_2}^{\omega_2}$$

then $\delta_3 = \delta_1 + \delta_2$ and $\omega_3 = \omega_1 + \omega_2$.

It is convenient, for digit sets of lower cardinality, to adopt a less cumbersome notation by replacing $\triangle_1$ by a , $\triangle_2$ by b , $\triangle_3$ by c , etc. Thus, $a^0 = \{0,1\}$, $a^1 = \{\bar{1},0\}$, $b^0 = \{0,1,2\}$, $b^1 = \{\bar{1},0,1\}$, $b^2 = \{\bar{2},\bar{1},0\}$, etc. The simplest symmetric digit set $b^1 = \{\bar{1},0,1\}$ will be of importance in sections to follow.

It may be noted that many of the structures for digital computer addition and subtraction, such as binary full adders and subtracters, conform to the rules of set addition; the symbol $\leftarrow$ is used to differentiate between inputs and outputs of such structures.

3. Basic derivations

For the investigation of the existence of distributivity, define

$$\triangle_{\delta_4}^{\omega_4} \leftarrow \triangle_{\delta_3}^{\omega_3} \times \left[\triangle_{\delta_1}^{\omega_1} + \triangle_{\delta_2}^{\omega_2} \right] \quad 3.1$$

$$\triangle_{\delta_5}^{\omega_5} \leftarrow \left[\triangle_{\delta_3}^{\omega_3} \times \triangle_{\delta_1}^{\omega_1} \right] + \left[\triangle_{\delta_3}^{\omega_3} \times \triangle_{\delta_2}^{\omega_2} \right] \quad 3.2$$

in which, for example, the product set $\triangle_{\delta_3}^{\omega_3} \times \triangle_{\delta_1}^{\omega_1}$ is found by

$$\{-\omega_3, -\omega_3 + 1, \dots, \bar{1}, 0, 1, \dots, \delta_3 - \omega_3\} \times \{-\omega_1, \dots, \bar{1}, 0, 1, \dots, \delta_1 - \omega_1\}$$

$$= \{-\max [\omega_1 (\delta_3 - \omega_3), \omega_3 (\delta_1 - \omega_1)], \dots, \bar{1}, 0, 1, \dots, \max [\omega_1 \omega_3, (\delta_1 - \omega_1)(\delta_3 - \omega_3)]\}$$

Note that, in general, a product set is not necessarily a digit set, since not all consecutive values may be included. For example, the product set $\{0, 1, 2\} \times \{0, 1, 2\} = \{0, 1, 2, 4\}$.

The definition of distributivity employed here is that the most negative and the most positive values of the sets $\triangle_{\delta_4}^{\omega_4}$ and $\triangle_{\delta_5}^{\omega_5}$ shall be identical, or that $\omega_4 = \omega_5$ and $\delta_4 - \omega_4 = \delta_5 - \omega_5$. 3.3

It is easy to determine that

$$\omega_4 = \max [\omega_3 (\delta_1 - \omega_1 + \delta_2 - \omega_2), (\delta_3 - \omega_3)(\omega_1 + \omega_2)] \quad 3.4$$

$$\omega_5 = \max [\omega_3 (\delta_1 - \omega_1), \omega_1 (\delta_3 - \omega_3)] + \max [\omega_3 (\delta_2 - \omega_2), \omega_2 (\delta_3 - \omega_3)] \quad 3.5$$

$$\delta_4 - \omega_4 = \max [\omega_3 (\omega_1 + \omega_2), (\delta_3 - \omega_3)(\delta_1 - \omega_1 + \delta_2 - \omega_2)] \quad 3.6$$

$$\delta_5 - \omega_5 = \max [\omega_3 \omega_1, (\delta_3 - \omega_3)(\delta_1 - \omega_1)] + \max [\omega_3 \omega_2, (\delta_3 - \omega_3)(\delta_2 - \omega_2)] \quad 3.7$$

4. Derivation of Some Specific Conditions for Distributivity

4.1 Introduction

The three specific conditions which appear to be of greatest practicality will be derived first. These are that distributivity exists if any two of the three digit sets $\triangle_{\delta_1}^{\omega_1}$, $\triangle_{\delta_2}^{\omega_2}$, or $\triangle_{\delta_3}^{\omega_3}$ are symmetric, or if $\triangle_{\delta_1}^{\omega_1}$ and $\triangle_{\delta_2}^{\omega_2}$ are both normalized or both negative normalized, or if $\triangle_{\delta_3}^{\omega_3}$ is either normalized or negative normalized. It is sufficient to show that symmetry applies simultaneously to $\triangle_{\delta_1}^{\omega_1}$ and $\triangle_{\delta_3}^{\omega_3}$, since commutativity of set addition then implies that the result will hold for $\triangle_{\delta_2}^{\omega_2}$ and $\triangle_{\delta_3}^{\omega_3}$.

4.2 Normalized Digit Sets

For a normalized digit set (See section 2) $\omega = 0$; therefore by substitution of $\omega_1 = \omega_2 = 0$ in equations 3.4 to 3.7, one obtains

$$\begin{aligned}\omega_4 &= \omega_3(\delta_1 + \delta_2) & \omega_5 &= \omega_3 \delta_1 + \omega_3 \delta_2 \\ \delta_4 - \omega_4 &= (\delta_3 - \omega_3)(\delta_1 + \delta_2) & \delta_5 - \omega_5 &= (\delta_3 - \omega_3)\delta_1 + (\delta_3 - \omega_3)\delta_2\end{aligned}$$

Therefore $\omega_4 = \omega_5$ and $\delta_4 - \omega_4 = \delta_5 - \omega_5$

Similarly by substitution of $\omega_3 = 0$, one obtains

$$\begin{aligned}\omega_4 &= \delta_3(\omega_1 + \omega_2) & \omega_5 &= \omega_1 \delta_3 + \omega_2 \delta_3 \\ \delta_4 - \omega_4 &= \delta_3[(\delta_1 - \omega_1) + (\delta_2 - \omega_2)] & \delta_5 - \omega_5 &= \delta_3(\delta_1 - \omega_1) + \delta_3(\delta_2 - \omega_2),\end{aligned}$$

and again $\delta_4 - \omega_4 = \delta_5 - \omega_5$.

4.3 Symmetric Digit Sets

For a symmetric digit set, $\delta = 2\omega$. By substitution of $\delta_1 = 2\omega_1$ and $\delta_2 = 2\omega_2$ in equations 3.4 to 3.7 one obtains:

$$\begin{aligned}\omega_4 &= (\omega_1 + \omega_2) \max [\omega_3, \delta_3 - \omega_3] \\ \omega_5 &= \omega_1 \max [\omega_3, \delta_3 - \omega_3] + \omega_2 \max [\omega_3, \delta_3 - \omega_3] \\ \delta_4 - \omega_4 &= (\omega_1 + \omega_2) \max [\omega_3, \delta_3 - \omega_3] \\ \delta_5 - \omega_5 &= \omega_1 \max [\omega_3, \delta_3 - \omega_3] + \omega_2 \max [\omega_3, \delta_3 - \omega_3],\end{aligned}$$

which implies that $\delta_4 = 2\omega_4 = \delta_5 = 2\omega_5$.

By substitution of $\delta_1 = 2\omega_1$ and $\delta_3 = 2\omega_3$, one obtains

$$\begin{aligned}\omega_4 &= \omega_3 [\omega_1 + \max (\omega_2, \delta_2 - \omega_2)] \\ \omega_5 &= \omega_3 \omega_1 + \omega_3 \max [\omega_2, \delta_2 - \omega_2]\end{aligned}$$

$$\delta_4 - \omega_4 = \omega_3 [\omega_1 + \max(\omega_2, \delta_2 - \omega_2)]$$

$$\delta_5 - \omega_5 = \omega_3 \omega_1 + \omega_3 \max[\omega_2, \delta_2 - \omega_2]$$

which implies that $\delta_4 = 2\omega_4 = \delta_5 = 2\omega_5$

4.4 Negative Normalized Digit Sets

For a negative normalized digit set, $\delta = \omega$. If $\delta_1 = \omega_1$ and $\delta_2 = \omega_2$, substitution in equations 3.4 to 3.7 yields

$$\omega_4 = (\delta_3 - \omega_3)(\omega_1 + \omega_2) \quad \omega_5 = \omega_1 (\delta_3 - \omega_3) + \omega_2 (\delta_3 - \omega_3)$$

$$\delta_4 - \omega_4 = \omega_3 (\omega_1 + \omega_2) \quad \delta_5 - \omega_5 = \omega_3 \omega_1 + \omega_3 \omega_2$$

Therefore, $\omega_4 = \omega_5$, and $\delta_4 - \omega_4 = \delta_5 - \omega_5$.

Similarly, for $\delta_3 = \omega_3$, the results are:

$$\omega_4 = \omega_3 [(\delta_1 - \omega_1) + (\delta_2 - \omega_2)] \quad \omega_5 = \omega_3 (\delta_1 - \omega_1) + \omega_3 (\delta_2 - \omega_2)$$

$$\delta_4 - \omega_4 = \omega_3 (\omega_1 + \omega_2) \quad \delta_5 - \omega_5 = \omega_3 \omega_1 + \omega_3 \omega_2$$

and again, $\omega_4 = \omega_5$ and $\delta_4 - \omega_4 = \delta_5 - \omega_5$.

4.5 Other conditions for distributivity

A number of other conditions for distributivity have been found, primarily by a systematic search of examples, after elimination of cases previously found. These include:

$$a) \quad \triangle_{\delta_1}^{\omega_1} = \triangle_{\delta_2}^{\omega_2}$$

$$b) \quad \triangle_{\delta_1}^{\omega_1} = - \triangle_{\delta_3}^{\omega_3} \quad \text{if either } \omega_1 = 0 \text{ or } \omega_3 = 0$$

$$c) \quad \triangle_{\delta_1}^{\omega_1} = - \triangle_{\delta_3}^{\omega_3} \quad \text{if } \delta_1 - \omega_1 \geq \omega_1 \text{ and } (\delta_1 - \omega_1)(\delta_2 - \omega_2) \geq \omega_1 \omega_2$$

or $\omega_1 \geq \delta_1 - \omega_1$ and $\omega_1 \omega_2 \geq (\delta_1 - \omega_1)(\delta_2 - \omega_2)$

- d) $\delta_3 = 2\omega_3$ if $\delta_1 - \omega_1 > \omega_1$ and $\delta_2 - \omega_2 > \omega_2$
or $\omega_1 > \delta_1 - \omega_1$ and $\omega_2 > \delta_2 - \omega_2$
- e) $\delta_1 + \delta_2 = 2(\omega_1 + \omega_2)$ and $\omega_3 = 0$ or $\delta_3 = \omega_3$

It is anticipated that a continuation of the systematic search will reveal more examples, with increasingly complex conditions for their applicability. Whether the results will be merely an intellectual curiosity or of practical value is yet to be determined.

5. Representations of Symmetric Digit Sets

The results of section 4 indicate that, for array multipliers involving the use of negative numbers, consideration should be given to representations of symmetric digit sets of higher cardinality as combinations of symmetric digit sets of low cardinality. The purpose of this section is to show that this can be done, and to indicate how the desired combinations of digit sets of low cardinality can be obtained.

In particular, the simplest symmetric digit set is $\triangle_2^1 = b^1 = \{\bar{1}, 0, 1\}$. Table 5.1 indicates the representations for symmetric digit sets with $\delta = 2\omega$, for values of ω ranging from 1 to 16. The third column of Table 5.1 gives the representations, suitable for multiplication, which employ only the digit set b^1 . The table can be extended to arbitrarily high values of ω , by the following somewhat awkward procedure:

ω	$\triangle^{2\omega}$	minimal	multiplication
1	$\triangle^1_2$	b^1	b^1
2	$\triangle^2_4$	$2a^1 + b^0$	$b^1 + b^1$
3	$\triangle^3_6$	$2b^1 + b^1$	$2b^1 + b^1$
4	$\triangle^4_8$	$4a^1 + 2a^0 + b^0$	$2b^1 + b^1 + b^1$
5	$\triangle^5_{10}$	$4a^1 + 2b^0 + b^1$	$2b^1 + 2b^1 + b^1$
6	$\triangle^6_{12}$	$4b^1 + 2a^1 + b^0$	$4b^1 + b^1 + b^1$
7	$\triangle^7_{14}$	$4b^1 + 2b^1 + b^1$	$4b^1 + 2b^1 + b^1$
8	$\triangle^8_{16}$	$8a^1 + 4a^0 + 2a^0 + b^0$	$4b^1 + 2b^1 + b^1 + b^1$
9	$\triangle^9_{18}$	$8a^1 + 4a^0 + 2b^0 + b^1$	$4b^1 + 2b^1 + 2b^1 + b^1$
10	$\triangle^{10}_{20}$	$8a^1 + 4b^0 + 2a^1 + b^0$	$4b^1 + 4b^1 + b^1 + b^1$
11	$\triangle^{11}_{22}$	$8a^1 + 4b^0 + 2b^1 + b^1$	$4b^1 + 4b^1 + 2b^1 + b^1$
12	$\triangle^{12}_{24}$	$8b^1 + 4a^1 + 2a^0 + b^0$	$8b^1 + 2b^1 + b^1 + b^1$
13	$\triangle^{13}_{26}$	$8b^1 + 4a^1 + 2b^0 + b^1$	$8b^1 + 2b^1 + 2b^1 + b^1$
14	$\triangle^{14}_{28}$	$8b^1 + 4b^1 + 2a^1 + b^0$	$8b^1 + 4b^1 + b^1 + b^1$
15	$\triangle^{15}_{30}$	$8b^1 + 4b^1 + 2b^1 + b^1$	$8b^1 + 4b^1 + 2b^1 + b^1$
16	$\triangle^{16}_{32}$	$16a^1 + 8a^0 + 4a^0 + 2a^0 + b^0$	$8b^1 + 4b^1 + 2b^1 + b^1 + b^1$

Table 5.1 Representations of Symmetric Digit Sets

a) First observe the predictable pattern of representations for values of $\omega = 2^\kappa$, where κ is an integer.

κ	ω	multiplicative representation
0	1	b^1
1	2	$b^1 + b^1$
2	4	$2b^1 + b^1 + b^1$
3	8	$4b^1 + 2b^1 + b^1 + b^1$
4	16	$8b^1 + 4b^1 + 2b^1 + b^1 + b^1$

b) Write the value of ω as a binary number. For example, if $\omega = 9$, the binary equivalent is 1001. Then associate all 0's with the 1 immediately to their left; the association for 1001 is (100)(1), indicating that 1001 should be regarded as $2 \times 4 + 1$. The representation for $\omega = 9$ is then a combination of the representation for 4 multiplied by 2 plus the representation for 1, or $2(2b^1 + b^1 + b^1) + b^1 = 4b^1 + 2b^1 + 2b^1 + b^1$. Similarly, if $\omega = 10100011 = 163$, the associations and corresponding weights are

$$\begin{array}{cccccccc} 128 & 64 & 32 & 16 & 8 & 4 & 2 & 1 \\ (1 & 0) & (1 & 0 & 0 & 0) & (1) & (1) \end{array}$$

and the representation is

$$64 (b^1 + b^1) + 4 (4b^1 + 2b^1 + b^1 + b^1) + 2b^1 + b^1.$$

6. Implications on the Design of Array Multipliers for Radix Complement Representations

A radix complement representation can be transformed into a symmetric representation by a generalization of the Booth (or differentiating) recoding. Let $y = -2^5y_5 + 2^4y_4 + 2^3y_3 + 2^2y_2 + 2y_1 + y_0$. Then $2y = -2^6y_5 + 2^5y_4 + 2^4y_3 + 2^3y_2 + 2^2y_1 + 2y_0 + y_{-1}$, with $y_{-1} = 0$. Also $-2^5y_5 = -2^6y_5 + 2^5y_5$.

$$y = 2y - y = 2^6 (-y_5 + y_5) + 2^5 (y_4 - y_5) + 2^4 (y_3 - y_4) \\ + 2^3 (y_2 - y_3) + 2^2 (y_1 - y_2) + 2 (y_0 - y_1) + (y_{-1} - y_0)$$

In the binary form shown, the resulting recoding is the Booth or differentiating recoding, and transforms each binary digital position into a redundant symmetric digit set $b^1 = \{\bar{1}, 0, 1\}$.

The above formulation may also be written as

$$2^4 (-2y_5 + y_4 + y_3) + 2^2 (-2y_3 + y_2 + y_1) + 2^0 (-2y_1 + y_0 + y_{-1})$$

Each radix 4 digit is then a member of the symmetric digit set $d^2 = \{\bar{2}, \bar{1}, 0, 1, 2\}$.

From Table 5.1, note that $d^2 = 2a^1 + b^0 = b^1 + b^1$. Note that in digital position i (i even), the set $2a^1 = \{\bar{2}, 0\}$ is represented by $-2y_{i+1}$ and the digit set $b^0 = \{0, 1, 2\}$ is the sum $y_i + y_{i-1}$.

From a theoretical point of view, the use of symmetric digit sets is apparently advantageous. First, distributivity is preserved, as indicated in section 4.3. Second, for a normalized digit set of diminished cardinality δ (δ even), the diminished cardinality of the product set is δ^2 . In contrast, for a symmetric digit set with $\delta = 2\omega$, the product set ranges from $-\delta^2/4$ to $\delta^2/4$, with diminished cardinality $\delta^2/2$.

For the purposes of this paper, an $n \times n$ array multiplier will be considered to consist of two parts; an array of elementary multipliers, and a summing network which converts the weighted outputs of the multiplier array to the desired weighted binary product.

Current practice is to use an array of n^2 AND gates for the elementary multipliers, with each AND gate performing an elementary multiplication of one of the types $\{\bar{1}, 0\} \times \{0, \bar{1}\} = \{0, 1\}$, or $\{\bar{0}, 1\} \times \{0, 1\} = \{\bar{0}, 1\} \times \{\bar{1}, 0\} = \{0, 1\}$. The diminished cardinality of the weighted sum of the AND gate array will always be $2^{2n} - 2^{n+1} + 1$.

regardless of the weights assigned to the individual digits of the multiplier or multiplicand, as the examples of section 1 indicate. The assumption is made (incorrectly), for the design of the summing network, that the AND gate outputs are independent of one another. Since the product requires 2η digits, with diminished cardinality $2^{2\eta} - 1$, the design parameters (See ref. 8.5) for the summing network are $\lambda_{in} = \eta^2$, $\delta_{myth} = 2^{\eta+1} - 2$, $\lambda_{myth} = \eta$, $\lambda_{out} = 2\eta$, so the cost of the summing network is $\eta^2 + \eta - 2\eta = \eta(\eta - 1)$ full adders, of which η can be reduced to half adders.

Although the use of symmetric digit sets appears to be theoretically attractive, the recoding procedures necessary to achieve them increase the information content. For example, for the Booth recoding, each digital position requires a type b^1 digit set, and each element of the multiplier array performs the operation $b^1 \leftarrow b^1 \times b^1$, at a cost roughly equivalent to a half adder. The information from each of the η^2 elementary multipliers is also doubled, which increases the cost of the summing network as well. Thus, use of the Booth recoding is clearly not cost effective.

Use of the radix four symmetric digit set $2a^1 + b^0$ requires the use of $\eta^2/4$ elementary multipliers of the form $4a^1 + 2a^0 + b^0 \leftarrow (2a^1 + b^0) \times (2a^1 + b^0)$. The output requires 4 bits, so that the information presented to the summing network is $4(\eta^2/4) = \eta^2$, which is the same as that of the conventional array. Thus the summing network should require roughly the same amount of hardware as the conventional array, but each elementary multiplier is more complicated than the four AND gates of the conventional array that it replaces.

It is also possible to compare the array version of the serial multiplier used in Illiac II and Illiac III. The multiplicand $x = -2^{\eta-1}x_{\eta-1} + 2^{\eta-2}x_{\eta-2} + \dots + 2^jx_j + \dots + 2^0x_0$ was left in conventional form and the multiplier y was recoded radix 4, with each digit of the form $d^2 = \{\bar{2}, \bar{1}, 0, 1, 2\}$. Thus $y = 2y - y = \sum_{i=0}^{\frac{1}{2}(\eta-2)} (-2y_{2i+1} + y_{2i} + y_{2i-1})$,

with $y_{-1} = 0$. The recoded digit of the form d^2 controls a conditional complementing and doubling circuit. This requires $\eta/2$ circuits of the form

$$+1 \quad q_{2i} = \overline{y}_{2i+1} (y_{2i} + y_{2i-1}) \quad 6.1$$

$$+2 \quad r_{2i} = \overline{y}_{2i+1} y_{2i} y_{2i-1} \quad 6.2$$

$$-2 \quad s_{2i} = y_{2i+1} \overline{y}_{2i} \overline{y}_{2i-1} \quad 6.3$$

$$-1 \quad t_{2i} = y_{2i+1} (y_{2i} + y_{2i-1}) \quad 6.4$$

Each of the $\frac{1}{2}\eta^2$ elements of the array requires circuits of the form

$$u = x_j q_{2i} v \overline{x}_j t_{2i} \quad j=0, \dots, \eta-1, i=0, \dots, \frac{\eta-2}{2} \quad 6.5$$

$$v = x_{j+1} r_{2i} v \overline{x}_{j+1} s_{2i} \quad 6.6$$

The process of complementation is completed by adding $\sum_{i=0}^{\frac{1}{2}(\eta-2)} 4^i (2s_{2i} + t_{2i})$.

The input to the summing network therefore consists of $\frac{\eta}{2}(\eta + 1)$ digit sets of type b, so $\beta_{in} = \eta/2(\eta + 1)$, $\alpha_{in} = 0$, $\lambda_{in} = \eta(\eta + 1)$ and $\alpha_{out} = 2\eta$, $\beta_{out} = 0$, $\lambda_{out} = 2\eta$. The approximate cost of the summing network is therefore $\eta(\eta - 1)$ carry generators and $\frac{1}{2}\eta(\eta - 3)$ half adders, as compared to $\eta(\eta - 1)$ carry generators and $\eta(\eta - 2)$ half adders for the summing network when the conventional array of η^2 AND gates is used. However, the radix four recoding requires $\frac{1}{2}\eta^2$ elements of type 6.5 and 6.6 and $\frac{1}{2}\eta$ elements of the types 6.1 to 6.4 for the elementary array multiplication, so the overall costs are approximately equal.

7. Conclusions

For conventional representations of numbers, the results obtained are disappointing. For the standard $\eta \times \eta$ array of AND gates, the diminished cardinality of the array output remains invariant when the weights of some of the

digits are made negative, and no advantage can be taken of the reduced cardinality which theoretically could be achieved.

Attempts to convert from conventional to symmetric digit sets are self-defeating, because of the increase in information content and because of the increased complexity of both the multiplier array and the summing network.

The technique of recoding the multiplier radix 4, with conditional complementation and doubling, appears to deserve further study in detail, since the order of magnitude calculations of section 6 indicate that it is cost competitive with the conventional $n \times n$ AND gate array.

For signed-digit arithmetic, the results obtained reinforce the desirability of using symmetric digit sets. Not only are symmetric digit sets advantageous for subtraction, since x and $-x$ are always members of the set, but the results indicate that symmetric digit sets are most efficient for multiplication.

For symmetric array multipliers, the tradeoff between the cost of the elementary multiplier array and the cost of the summing network deserves investigation. It appears to be advantageous to use $n^2/2$ elementary multipliers of the form $2a^1 + b^0 \leftarrow b^1 \times (2a^1 + b^0)$ rather than n^2 multipliers of the form $b^1 \leftarrow b^1 \times b^1$, because of the reduced costs of the summing network. Use of $n^2/4$ elementary multipliers of the form $4a^1 + 2a^0 + b^0 \leftarrow (2a^1 + b^0) \times (2a^1 + b^0)$ should also be investigated.

Note that the designs of elementary multipliers must be logical designs. For example $b^1 \times (2a^1 + b^0)$ should not be designed as $(b^1 \times 2a^1) + (b^1 \times b^0)$, since distributivity does not hold. The results obtained thus indicate which algebraic manipulations are permissible for efficient multiplier design.

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